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DYNAMICAL SYMMETRY AND SPIN WAVES OF ISOTROPIC ANTIFERROMAGNET

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Noncompact group $\prod_{\vec{k}} \otimes [SU(1,1)]_{\vec{k}}$ is shown to be the dynamical symmetry group of a linearized isotropic Heisenberg antiferromagnet. Eigenvalues and eigenstates corresponding to the spin waves are constructed. A group theoretical interpretation of the Bogolubov transformation as a hyperbolic rotation in the algebra space of the dynamical group is established. In the framework of the coherent states technique it is shown that the corresponding classical dynamics of the model is described by the harmonic oscillations on the Lobachevsky plane.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Динамическая симметрия и спиновые волны в изотропном антиферромагнетике

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Показано, что динамической группой симметрии линеаризованного изотропного антиферромагнетика является некомпактная группа $\prod_{k} \otimes SU(1,1)_{k}$. Построены дискретный спектр и собственные состояния, соответствующие спиновым волнам. Установлен теоретико-групповой смысл преобразования Боголюбова как гиперболического вращения в пространстве алгебры динамической группы. Методом когерентных состояний показано, что соответствующее классическое движение системы описывается гармоническими колебаниями в плоскости Лобачевского.

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1. As is well-known /1/, the dynamical symmetry group has appeared in the particle physics when various multiplets of particles were tried to be joined in a one irreducible representation of some noncompact group. After that

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they found applications in a number of one-particle problems (hydrogen atom, etc.) and many particle problems (superfluidity, etc.) [2-4]. In the case of the ordinary symmetry groups the Hamiltonian of the system commutes with all their generators. It means that energy degenerated states are transformed via the group symmatry representations. This is not the case for the dynamical symmetry group when the Hamiltonian of the system belongs simply to the group algebra. It is therefore named the spectrum generating algebra.

In some many-particle problems when linearized the Hamiltonian of the system may be written through the generators of some dynamical symmetry group. Then, solution of the quantum mechanical eigenvalue problem comes to the solution of the proper group theoretical problem, i.e., to looking for the irreducible representations of the dynamical group. In addition to the complete information about the spectra of the problem, the dynamical group makes it possible i) to construct the natural coherent states for the initial quantum system, ii) to represent the Green functions via the path integral over this states and iii) to describe classical behaviour of the system [7,11].

The theory of magnetism is especially interesting in this respect since there is a quantum microscopic theory based on the Heisenberg model on the one hand and the macroscopic theory of magnetism governed by the Landau-Lifshitz equations [5] on the other. Sometimes one can establish such a correspondence, for example, for isotropic ferromagnet [6] but not yet for the antiferromagnet and some models of anisotropic ferromagnet. As a first step in this direction one can construct the dynamical group and the coherent states of the proper spin models.

In the present communication the dynamical symmetry group of a linearized two sublattices Heisenberg antiferromagnet will be found as well as (using its irreducible unitary representations) the spectrum and eigenstates of the spin waves. Then the coherent states and Green function will be introduced. In conclusion we show that the classical dynamics of the linearized antiferromagnet as well as a nearly ideal superfluid Bose gas is described by the harmonic oscillators on the Lobachevsky plane.

2. Supposing the interaction is between the nearest neighbours only then for the Hamiltonian of two sublattices antiferromagnet we have [8]:

$$\mathcal{H} = \sum_{i\in A, j\in B} [\frac{1}{2}(S_i^+ S_j^+ + h.c.) - S_i^z S_j^z]. \quad (1)$$

Here, up-spins form sublattice A and down-spins form sublattice B. In the ground state every up-spin is in closed neighbourhood with every down-spin and vice versa.

In a linear approximation it follows from (1) that:

$$\mathcal{H}_{lin} = \frac{1}{2}\sum_i \sum_{j(i)} [s(a_i^+ a_j^+ + h.c.) - s^2 + s(a_i^+ a_i + a_j^+ a_j)] \quad (2)$$

or in the momentum representation[8]:

$$\mathcal{H}_{lin} = -\frac{z}{2} Ns^2 + sz\sum_{\vec k} a_{\vec k}^+ a_{\vec k} + \frac{s}{2}\sum_{\vec k,\vec\delta} \cos\vec k\vec\delta\,(a_{\vec k}^+ a_{-\vec k}^+ + a_{\vec k} a_{-\vec k}), \quad (3)$$

where z is the number of the near neighbours, $\vec\delta$ is the lattice vector, $a_{\vec k}^+ (a_{\vec k})$ is the Bose operator of the creation (annihilation) of a magnon with momentum $\vec k$ and

$$[a_{\vec k}, a_{\vec k'}^+] = \delta_{\vec k \vec k'}, \quad [a_{\vec k}, a_{\vec k'}] = [a_{\vec k}^+, a_{\vec k'}^+] = 0.$$

Let us following[3] introduce the operators:

$$J_1^{(\vec k)} = -\frac{1}{2}(a_{\vec k}^+ a_{-\vec k}^+ + a_{\vec k} a_{-\vec k}), \quad J_2^{(\vec k)} = \frac{i}{2}(a_{\vec k}^+ a_{-\vec k}^+ - a_{\vec k} a_{-\vec k}),$$

$$J_3^{(\vec k)} = \frac{1}{2}(a_{\vec k}^+ a_{\vec k} + a_{-\vec k}^+ a_{-\vec k} + 1) \quad (4)$$

generating the algebra of the $[SU(1,1)]_{\vec k}$ group:

$$[J_1^{(\vec k)}, J_2^{(\vec k)}] = -iJ_3^{(\vec k)}, \quad [J_2^{(\vec k)}, J_3^{(\vec k)}] = iJ_1^{(\vec k)}, \quad [J_3^{(\vec k)}, J_1^{(\vec k)}] = iJ_2^{(\vec k)}. \quad (5)$$

In terms of this operators the Hamiltonian (3) assumes the form:

$$\mathcal{H}_{lin} = -\frac{z}{2} Ns^2 + s\sum_{\vec k} [zJ_3^{(\vec k)} - J_1^{(\vec k)} \sum_{\vec\delta} \cos\vec k\vec\delta - \frac{z}{2}]. \quad (6)$$

The quantum mechanical problem is to solve the following eigenvalue problem:

$$\mathcal{H}_{lin} |\Psi_n\rangle = E_n |\Psi_n\rangle. \quad (7)$$

To construct the discrete spectrum of this problem let us perform the hyperbolic rotation by the angle $\theta_{\vec k}$ over the axis $J_2^{(\vec k)}$:

$$\mathcal{H}_{lin} \to \bar{\mathcal{H}}_{lin} = R\,\mathcal{H}_{lin} R^{-1}, \quad (8)$$

where $R = \prod_{\vec{k}} R(\theta_{\vec{k}})$, $R(\theta_{\vec{k}}) = \exp\{-iJ_2^{(\vec{k})}\theta_{\vec{k}}\}$. It is important to emphasize that transformation (8) is a group theoretical analogue of the well-known Bogolubov u-v transformation [10]. As a result we have

$$\bar{\mathcal{H}}_{lin} = -\frac{z}{2} N s^2 + s \sum_{\vec{k}} \left(\sqrt{z^2 - (\sum_{\vec{\delta}} \cos \vec{k}\vec{\delta})^2}\, J_3^{(\vec{k})} - \frac{z}{2}\right), \qquad (9)$$

where $\mathrm{th}\,\theta_{\vec{k}} = \frac{1}{z} \sum_{\vec{\delta}} \cos \vec{k}\vec{\delta}$.

The Casimir operator of the $[SU(1,1)]_{\vec{k}}$ group is:

$$C_{\vec{k}} = (J_3^{(\vec{k})})^2 - (J_1^{(\vec{k})})^2 - (J_2^{(\vec{k})})^2 = \frac{1}{4}(\Delta^2_{\vec{k}} - 1), \qquad (10)$$

where $\Delta_{\vec{k}} = a^+_{\vec{k}} a_{\vec{k}} - a^+_{-\vec{k}} a_{-\vec{k}}$.

The only possible irreducible unitary representation $\prod_{\vec{k}} \otimes D^+(j_{\vec{k}})$ of the $\prod_{\vec{k}} \otimes [SU(1,1)]_{\vec{k}}$ group corresponding to bound below spectrum is as follows:

$$J_3^{(\vec{k})} |n_{\vec{k}}\rangle = (n_{\vec{k}} + \sigma_{\vec{k}}) |n_{\vec{k}}\rangle, \qquad (11)$$

where $|n_1, n_2, \ldots, n_{\vec{k}}, \ldots\rangle = \prod_{\vec{k}} |n_{\vec{k}}\rangle$, $n_{\vec{k}} = 0,1,2,\ldots$; $\sigma_{\vec{k}} = \frac{1}{2}(1 + |\Delta_{\vec{k}}|) = -j_{\vec{k}}$. For energy and eigenstates of the Hamiltonian (9) from (7), (8) and (9) it follows that:

$$E_{n_1,\ldots,n_{\vec{k}},\ldots} = \sum_{\vec{k}} \left(n_{\vec{k}} + \frac{1}{2} + \frac{|\Delta_{\vec{k}}|}{2}\right) E_{\vec{k}} - \frac{z}{2} N s(s+1), \qquad (12)$$

where $E_{\vec{k}} = s\sqrt{z^2 - (\sum_{\vec{\delta}} \cos \vec{k}\vec{\delta})^2}$ is an antiferromagnon excitation spectrum

$$|\Psi(n_1, \ldots, n_{\vec{k}}, \ldots)\rangle = \prod_{\vec{k}} \otimes \sum_{m_{\vec{k}}} S^{j_{\vec{k}}}_{m_{\vec{k}} n_{\vec{k}}}(g_{\vec{k}}) |m_{\vec{k}}\rangle, \qquad (13)$$

where

$$g_{\vec{k}} = \begin{pmatrix} a_{\vec{k}} & \beta_{\vec{k}} \\ \bar{\beta}_{\vec{k}} & \bar{a}_{\vec{k}} \end{pmatrix} \in [SU(1,1)]_{\vec{k}},$$

$a_{\vec{k}} = \mathrm{ch}\frac{\theta_{\vec{k}}}{2}$, $\beta_{\vec{k}} = \mathrm{sh}\frac{\theta_{\vec{k}}}{2}$, $S^{j_{\vec{k}}}_{m_{\vec{k}} n_{\vec{k}}}(g_{\vec{k}})$ are the finite matrix elements of the unitary irreducible representations corresponding to the group element $g_{\vec{k}} \in [SU(1,1)]_{\vec{k}}$ [3]. In particular, for the ground state ($n_{\vec{k}} = 0$, $\Delta_{\vec{k}} = 0$, $j_{\vec{k}} = -1/2$) one gets:

$$E_{0,\dots,0,\dots} = \frac{s}{2}\sum_{\vec{k}}\sqrt{z^2-(\sum_{\vec{\delta}}\cos\vec{k}\vec{\delta})^2} - \frac{z}{2}Ns(s+1), \tag{14}$$

$$|\Psi(0,\dots,0,\dots)> = \prod_{\vec{k}} \operatorname{sech}\frac{\theta_{\vec{k}}}{2}\exp\{-\sum_{\vec{k}} \operatorname{th}\frac{\theta_{\vec{k}}}{2} a^+_{\vec{k}} a^+_{-\vec{k}}\}|0>, \tag{15}$$

where

$$a_{\vec{k}}|0> = 0, \quad \operatorname{th}\frac{\theta_{\vec{k}}}{2} = \frac{z-\sqrt{z^2-(\sum_{\vec{\delta}}\cos\vec{k}\vec{\delta})^2}}{\sum_{\vec{\delta}}\cos\vec{k}\vec{\delta}}.$$

This result, as it might be expected coincides, with the one getting via the Bogolubov transformation [8].

For the linear spin chain (z = 2) we have the following spectrum of an antiferromagnon excitation

$$\mathcal{E}(k) = 2s|\sin ka|, \tag{16}$$

which is well-known from the Anderson solution of an isotropic antiferromagnet model [12]. The eigenvector of the ground state of the system is

$$|\Psi(0,\dots,0,\dots)> = \prod_k \operatorname{sech}\frac{\theta_k}{2}\exp\{-\sum_k \operatorname{th}\frac{\theta_k}{2}a^+_k a^+_{-k}\}|0>, \tag{17}$$

where $\operatorname{th}\frac{\theta_k}{2} = \frac{1-|\sin ka|}{\cos ka}$.

Dynamical symmetry group allows one to construct the coherent states of the system and to represent its propagator via the path integral over these states [11]. In our case the propagator of the system can be presented in the form:

$$K(\zeta',\zeta;T) = e^{i\frac{z}{2}Ns^2T}\prod_{\vec{k}}\sum_{|\Delta_{\vec{k}}|}K_{|\Delta_{\vec{k}}|}(\zeta'_{\vec{k}},\zeta_{\vec{k}};T), \tag{18}$$

where

$$K_{|\Delta_{\vec{k}}|}(\zeta'_{\vec{k}},\zeta_{\vec{k}};T) = \int d\mu_{|\Delta_{\vec{k}}|}(\zeta_{\vec{k}})\exp\{\frac{i}{h}\int dt\,\frac{ih\sigma_{\vec{k}}}{1-|\zeta_{\vec{k}}|^2}(\zeta_{\vec{k}}\dot{\bar{\zeta}}_{\vec{k}} - \bar{\zeta}_{\vec{k}}\dot{\zeta}_{\vec{k}}) + s[\sqrt{z^2-(\sum_{\vec{\delta}}\cos\vec{k}\vec{\delta})^2}\,\sigma_{\vec{k}}\frac{1+|\zeta_{\vec{k}}|^2}{1-|\zeta_{\vec{k}}|^2} - \frac{1}{2}z]\}.$$

Here the $[SU(1,1)]_{\vec{k}}$ coherent states have been used [7,11]:

$$|\zeta_{\vec{k}},\sigma_{\vec{k}}> = (1-|\zeta_{\vec{k}}|^2)^{\sigma_{\vec{k}}}\exp\{\zeta_{\vec{k}}J_+^{(\vec{k})}\}|\sigma_{\vec{k}},0>, \tag{19}$$

where

$$J_+^{(\vec{k})} = J_1^{(\vec{k})} + iJ_2^{(\vec{k})}, \quad d\mu_{|\Delta_{\vec{k}}|}(\zeta_{\vec{k}}) = \frac{2\sigma_{\vec{k}}-1}{\pi} \frac{d^2\zeta_{\vec{k}}}{(1-|\zeta_{\vec{k}}|^2)^2},$$

$$\frac{<\zeta_{\vec{k}}|J_3^{(\vec{k})}|\zeta_{\vec{k}}>}{<\zeta'_{\vec{k}}|\zeta_{\vec{k}}>} = \sigma_{\vec{k}} \frac{1+\bar{\zeta}'_{\vec{k}}\zeta_{\vec{k}}}{1-\bar{\zeta}'_{\vec{k}}\zeta_{\vec{k}}}. \tag{20}$$

The equivalent classical problem determined through a path integral (18) is dealing with a curved phase space namely Lobachevsky plane. It follows from the classical action in eq.(18) that the corresponding classical motion for the quantum linearized antiferromagnet will look oscillator-like on the Lobachevsky plane with frequencies:

$$\omega_{\vec{k}} = \frac{s}{h}\sqrt{z^2 - (\sum_{\vec{\delta}} \cos\vec{k}\vec{\delta})^2}.$$

In one space dimension $\omega_k = \frac{2s}{h}|\sin ka|$ so we have the quasiclassical result:

$$h\omega_k = 2s|\sin ka|. \tag{21}$$

As is well-known, for spin s = 1/2 the Anderson's excitation spectrum [12] coincides with the exact one in the XY model [8] and differs from the exact result for "hole"-like excitation spectrum in the isotropic antiferromagnet [9,13] up to the coefficient $2/\pi$. It follows from model (21) that the frequency of classical motion on Lobachevsky plane doesn't depend on the choice of the coherent state related to the representation space of the dynamical symmetry group. This means that described by the linearized antiferromagnet equations the harmonic motion cannot reproduce the configurations related to the exact result for the quantum integrable system [13], and it is necessary to study more complex soliton-like configurations of the nonlinear classical equations corresponding to the isotropic antiferromagnet. One of the possibilities is considered in paper [14], where the coefficient in excitation spectrum (21) is determined by the density of the magnon condensate.

It is important to emphasize once more that a coincidence of the dynamical symmetry groups for a linearized antiferromagnet and a nearly ideal Bose gas of superfluid type makes it possible to conclude that a classical motion in both cases is determined by the harmonic oscillators on the Lobachevsky plane. As is shown in paper [14], a similar character of motion is given by a classical integrable model of Heisenberg magnet with the magnetization vector lying on the hyperboloid $SU(1,1)/U(1)$. This model is gauge equivalent to the repulsive nonlinear Schrödinger equation [15] that describes a nearly ideal Bose gas on the classical level. It should be noted that if the ordinary coherent states are used in our problem, one immediately encounters the difficulties of the mixing of modes with $+\vec{k}$ and $-\vec{k}$ momenta. This difficulties are overcome with the introduction of the $SU(1,1)$ coherent states.

Thus one can say that the ground state of an antiferromagnet as well as of a superfluid Bose gas [10] may be related to the magnon condensation in momentum space.

In conclusion we remarked that the above consideratión may be applied to a number of problems of the magnetism theory. Among them there is a spin-wave theory providing an adequate description of the low-energy magnetic excitations in materials that can be described by a Heisenberg exchange Hamiltonian. When the boson Holstein-Primakoff representations are used,the corresponding dynamical symmetry groups are noncompact $\bigotimes_{\vec{k}} [SU(1,1)]_{\vec{k}}$. Related to the spin-wave classical motion following from the path integral in $SU(1,1)$ coherent states representation is the harmonic motion on the Lobachevsky plane. There are, for example, the spin waves in the anisotropical XXZ antiferromagnet, those in the ferromagnet with dipole interaction, in ferrimagnets and in the easy-plane anisotropical ferromagnet. It is important to note that for the spin $s = 1/2$ in one dimension there exist an exact Jordan-Wigner transformation from the Pauli to the Fermi operators. Related dynamical group is now compact $\bigotimes_{\vec{k}} [SU(2)]_{\vec{k}}$, and the classical motion is the harmonic oscillations on the sphere S^2. There are for example the XY ferromagnet as well as XXX ferromagnet with spin $s = 1/2$. These results with some other lattice models will be published elsewhere.

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